Comparison of a Direct and a Vector Potential Integral Equation Method for the Computation of Eddy Currents

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Abstract — This paper deals with the numerical computation of eddy current problems by means of the volume integral equation method with edge element based shape functions. Two techniques are presented. One allows for the direct computation of the eddy current density. The other one demands the derivation of the electric vector potential to obtain the eddy current density. The differences between these methods are shown, especially the different gauging techniques needed. The efficiency and the accuracy of both methods are examined.

I. INTRODUCTION

The indirect electric vector potential method has become quite popular in the recent years and has been developed by Rubinacci, Albanese, and others in the late eighties [1]. It is suitable for a wide field of applications, such as error detection, shape identification or in the design process of novel devices.

The novel direct method is introduced to give a suitable alternative to the established method. It is described in detail in [2].

II. PROBLEM DESCRIPTION

Eddy currents in non-magnetic, electrically conductive materials are considered in the time domain. Ohm's law and the induction law lead to the following equation

$$\frac{1}{\kappa}\boldsymbol{J}(\boldsymbol{r})+\frac{\partial}{\partial t}\frac{\mu_0}{4\pi}\int_V\frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V=-\frac{\partial}{\partial t}\boldsymbol{A}_s(\boldsymbol{r})\,. \qquad (1)$$

$\boldsymbol{A}_s$ is the source magnetic vector potential, $\boldsymbol{J}$ the eddy current density, κ the electrical conductivity, and μ_0 the magnetic field constant. Please note that the source magnetic vector potential $\boldsymbol{A}_s$ has to undergo a kind of gauging process depending on the chosen method.

III. SYSTEM ASSEMBLY FOR THE ELECTRIC VECTOR POTENTIAL METHOD

Introducing edge-element-based shape functions $\boldsymbol{N}_i$ and their rotations $\nabla\times\boldsymbol{N}_i$ and applying the Galerkin method, the problem describing system of linear equations (SLE) $\{U\}=[Z]\{T\}$ with $[Z]=[R]+\frac{\partial}{\partial t}[L]$ can be built of the terms

$$R_{ij}=\int_V\frac{1}{\kappa}\nabla\times\boldsymbol{N}_i(\boldsymbol{r})\cdot\nabla\times\boldsymbol{N}_j(\boldsymbol{r})\,\mathrm{d}V(N_E)\,, \qquad (2)$$

$$L_{ij}=\frac{\mu_0}{4\pi}\int_V\int_{V'}\frac{\nabla\times\boldsymbol{N}_i(\boldsymbol{r}')\cdot\nabla\times\boldsymbol{N}_j(\boldsymbol{r})}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'\mathrm{d}V\,, \qquad (3)$$

and

$$U_i=-\frac{\partial}{\partial t}\int_V\boldsymbol{A}_s\cdot\nabla\times\boldsymbol{N}_i(\boldsymbol{r})\,\mathrm{d}V\,. \qquad (4)$$

The obtained solution of the SLE are the integrals of the electric magnetic vector potential $\boldsymbol{T}$ over the element edges of the discretized eddy current region. The eddy current density inside an element is obtained by derivation

$$\boldsymbol{J}(\boldsymbol{r})=\sum_{i=1}^{n}\int_V\nabla\times\boldsymbol{N}_i(\boldsymbol{r})T_i\mathrm{d}V\,. \qquad (5)$$

V is the element volume, $\nabla\times\boldsymbol{N}_i(\boldsymbol{r})$ the rotation of the edge shape function of edge i, and T_i the solution value of this edge.

IV. SYSTEM ASSEMBLY FOR THE DIRECT METHOD

Now, the edge-element-based shape functions $\boldsymbol{N}_i$ are used without a derivation. With application of the Galerkin method, the alternative SLE $\{U\}=[Z]\{I\}$ with $[Z]=[R]+\frac{\partial}{\partial t}[L]$ is then built of the terms

$$R_{ij}=\int_V\frac{1}{\kappa}\boldsymbol{N}_i(\boldsymbol{r})\cdot\boldsymbol{N}_j(\boldsymbol{r})\,\mathrm{d}V(N_E)\,, \qquad (6)$$

$$L_{ij}=\frac{\mu_0}{4\pi}\int_V\int_{V'}\frac{\boldsymbol{N}_i(\boldsymbol{r}')\cdot\boldsymbol{N}_j(\boldsymbol{r})}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'\mathrm{d}V\,, \qquad (7)$$

and

$$U_i=-\frac{\partial}{\partial t}\int_V\boldsymbol{A}_s\cdot\boldsymbol{N}_i(\boldsymbol{r})\,\mathrm{d}V\,. \qquad (8)$$

The solution vector $\{I\}$ of the SLE contains the integrals of the eddy current density over an edge. The eddy current density is obtained by

$$\boldsymbol{J}(\boldsymbol{r})=\sum_{i=1}^{n}\int_V\boldsymbol{N}_i(\boldsymbol{r})I_i\mathrm{d}V\,. \qquad (9)$$

Both methods require to satisfy the following conditions in the conducting domain Ω

$$\nabla\cdot\boldsymbol{J}=0 \qquad (10)$$

and on its boundary $\partial\Omega$

$$\boldsymbol{n}\cdot\boldsymbol{J}=0 \qquad (11)$$

with the outward normal vector $\boldsymbol{n}$ on the boundary.

The Conditions of equations (10) and (11) can be verified by the choice of suitable shape functions and topological means of network theory.

V. VERIFICATION OF THE BOUNDARY CONDITION IN THE VECTOR POTENTIAL METHOD

The current flux trough an element's face is determined by the sum of the surrounding edge integrals over $\boldsymbol{T}$. The other way round an edge integral over $\boldsymbol{T}$ corresponds to a current loop around this edge. Therefore, zeroing all edges at the boundary of the conducting domain assures the condition of equation (11).

Since current densities in all elements are built as a sum of loop currents around the edges, the condition of equation (10) is obviously fulfilled. If the elements faces are considered as elements of an electrical network, the means of the well-known loop current analysis can be applied. This is afforded by a tree cotree algorithm. All edges are divided into tree and cotree edges. The values of the edges at the domain boundary and the tree edges are zeroed. The inner cotree edges are treated according to equations (2) to (4). No other modifications are required. Thus, the edge integral values of the inner cotree edges are the unknowns of the SLE. Compared to the FEM, the number of unknowns is very small, since there are no unknowns in the air domain and many unknowns in the conducting region are eliminated [5].

VI. VERIFICATION OF THE BOUNDARY CONDITIONS IN THE DIRECT METHOD

In contrast to the vector potential method, the current flux is represented by the edges. The edges are considered as the element of an electrical circuit and the loop current analysis is applied. In order to avoid large loops that would lead to increasing computing times and an ill conditioned SLE, every loop is limited to the edges surrounding an element face. If the mesh consists of hexahedral elements of first order, the number of edges per loop is always four.

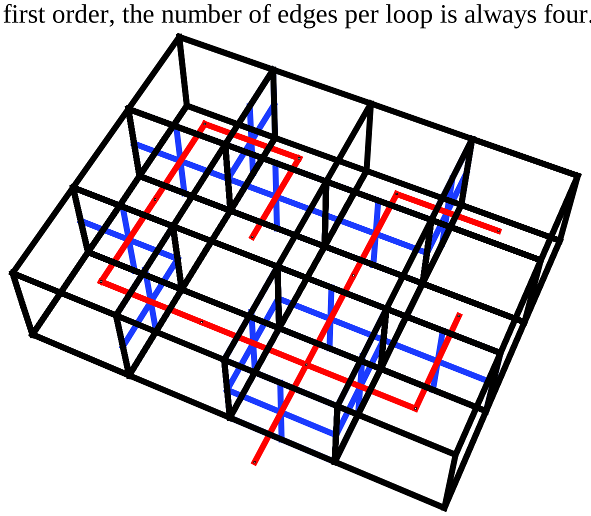

Figure 5: Tree (red) on the dual mesh, that detects the linearly dependent faces (blue cross).

Due to the application of loop currents, the conditions of equations (10) and (11) are fulfilled. The question now is how to detect the linear independent face loops. This is done according to the algorithm in chapter V. The centroids of the elements are considered as the vertexes of a dual mesh. A tree cotree decomposition is run on the dual mesh. The faces passed by the cotree edges are the linear independent faces, the closed loop integrals over the current density are the unknowns of the SLE (see fig. 1).

VII. REORDERING THE SYSTEM OF THE DIRECT METHOD

After detecting the linear independent loops, the SLE has to be assembled. Assumed that a SLE $\{U\}=[Z]\{I\}$ with the total number of edges as unknowns is still assembled according to equations (6) to (8), a SLE $\{U^*\}=[Z^*]\{I^*\}$ with the total number of loops as unknowns can be obtained by the following reordering scheme.

The information about the loops is stored in a non-quadratic (l,e)-matrix $[M\ ]$, $l=n-e+1$ is the number of loops, e the number of edges, and n the number of nodes. The entries M_{ij} of $[M\ ]$ are zero, if edge j is not part of loop i. If edge j is part of loop i, the entry is either one ore minus one, depending on the loop and edge direction.

Thus, the reordering scheme to obtain the system matrix $[Z^*]$ can easily be described by

$$[Z^*]=[M][Z]. \tag{12}$$

The RHS $\{U^*\}$ of the minimized SLE is then the product of the transposed of $[M\ ]$ and the RHS $\{U\}$ of the original SLE

$$\{U^*\}=[M]^T\{U\}. \tag{13}$$

The solution on the edges $\{I\}$ is obtained by the product of $[M\ ]$ and the loop solution $\{I^*\}$

$$\{I\}=[M]\{I^*\}. \tag{14}$$

The direct method showed a more stable behavior. The SLE is easier to solve.

VIII. CONCLUSION

The efforts to assemble the SLE of the direct method are slightly higher than for the assembly of the vector potential method, since all edges require an integration routine and the number of unknowns is bigger.

IX. REFERENCES

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